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A THREE-VALUED DOXASTIC LOGIC BASED ON KLEENE'S AND BOCHVAR'S IDEAS

Abstract. In this paper, I shall propose the construction of a three-valued logic of beliefs, which I call: LSB3_1 (short for: Three-valued Logic for a type of Strong Belief). I shall also state and prove the completeness of LSB3_1 with respect to a given semantics.

LSB3_1 is based on preformal assumptions and intuitions, which are stated in section 1. Section 2 includes the syntax and division of LSB3_1 statements into internal and external. Section 3 presents the semantics of LSB3_1, as well as a number of tautologies and non-tautological formulae in LSB3_1 with their intuitive interpretation. The axiomatic system for LSB3_1 and its comparison to Kleene's strong logic are provided in section 4. The completeness theorem for LSB3_1 is presented in section 5. I shall define the term *conjunctive normal form* and provide lemmas which lead to proving the reduction of the LSB3_1 language formulae before proving the completeness theorem.

Keywords: doxastic logic, strong belief, Kleene's strong three-valued logic, internal connectives, external connectives, completeness theorem

1. BASIC ASSUMPTIONS OF LSB3_1

As far as doxastic and static epistemic logics are concerned, a common division of the beliefs of a cognitive subject (henceforth: "agent") is drawn according to their power. According to such a division, we can distinguish beliefs that are certain from beliefs with weaker power such as assumptions and admissions. In dynamic doxastic and epistemic logics, the division into non-changeable and changeable beliefs seems more natural.

However, there are also logics which divide beliefs differently. For instance, Weingartner's KBA logic[1], H.J. Levesque's doxastic logic[2] and R. Fagin's and J. Halpern's Logic of Awareness and Logic of General Awareness[3]. In Weingartner's logic, beliefs are divided according to whether they amount to knowledge or not. In particular, he distinguishes between *weak belief* and *strong belief*. H.J. Levesque distinguishes between explicit beliefs and implicit beliefs to construct a logic in which the agent is not logically omniscient. R. Fagin and J. Halpern introduce the notion of an agent's awareness and define an explicit belief as a belief about α which the agent holds when s/he is aware of α.

In this paper I present LSB3_1, i.e. the logic of one kind of strong belief. LSB3_1 is built upon the division of beliefs about complex sentences according to the way they are developed by an agent.

I assume that there are two ways of developing beliefs concerning complex sentences. According to the first way, such beliefs are dependent on the degree of confidence of an agent's beliefs about component sentences. Following this way (this method) of belief construction, if the agent is not convinced that neither disjunct is true, then he is not convinced that the whole sentence is true. In other words, an agent is convinced that a disjunction is true only if s/he is convinced that the first disjunct is true or that the second disjunct is true. The aim of this paper is precisely to construct a logic of this kind of strong belief.

The other way of developing beliefs about a complex sentence proceeds without any in-depth analysis of its constituents. For instance, in such a case, we may be convinced of the truth of a complex sentence

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1 P. Weingartner, *Conditions of Rationality for the Concepts Belief, Knowledge and Assumption*, *Dialectica* **36(**1982), 243–263.

2 H.J. Levesque, *A logic of implicit and explicit belief*, in: Proc. Of AAAI-84, Austin 1984, 198–202.

3 R. Fagin, J. Halpern, *Belief, awareness and limited reasoning*, Artificial Intelligence 34(1988), 49–57.

even if we are not convinced as to which of its constituent sentences are true. By applying this method, a person A can be convinced that a person B is currently in one of two places, even if A is not convinced as to which of the two places B really is. In *Ein System der epistemischen Logik* Ho Ngoc Duc presented a three valued logic with two kinds of connectives: internal connectives and external connectives[4]. This logic, in my opinion, concerns the latter way of developing beliefs about complex sentences[5].

In this paper I present a logic of strong beliefs, which concern complex sentences and are developed by the agent on the basis of the degree of confidence of his beliefs about their component sentences. I distinguish three kinds of subjective attitudes of the agent:

- the agent's strong belief that a given situation occurs (that it is a fact),
- the agent's strong belief that a given situation does not occur (that it is not a fact),
- the agent's lack of strong belief concerning both cases – that is, whether a situation is a fact or not.

I choose my own construction of the many-valued logic of strong belief. It seems that even the three-valued logic of beliefs can comprise basic formal properties of expressions such as "the agent has a strong belief that…", or "the agent is certain that…".

In constructing LSB3_1, I use Kleene's strong three-valued logic[6] and Bochvar's idea of the division of logical formulae into internal and external[7]. From Bochvar, I also borrow the idea to construct the

4 Ngoc Duc Ho, *Ein System der epistemischen Logik*, in: *Philosophie Und Logik: Frege--Kolloquien 1989 Und 1991*, ed. W. Stelzner, De Gruyter, Berlin – New York 1993, 205–214.

5 Although Ngoc Duc Ho writes that his logic concerns knowledge, by which it seems that he means a justified opinion or belief, rather than knowledge. See M. Lechniak, *Wielowartościowość a pojęcia epistemiczne*, Roczniki Filozoficzne 54(2006)2, 379–384.

6 S.C. Kleene, *On a notation for ordinal numbers*, The Journal of Symbolic Logic 3(1938)4, 150–155.

7 A. Urquhart, *Basic Many-Valued Logic*, in: *Handbook of Philosophical Logic*, vol.2, eds. D.M. Gabbay, F. Guenthner, Kluwer Academic Publishers, Dordrecht 2001, 253.

logic of an agent's various attitudes to beliefs towards propositions on the assumption that certain objective situations are facts (or that they are not facts).

The internal level expresses an agent's attitudes to beliefs, which are the external – or formal – counterparts to objective propositions (or situations corresponding to such propositions). As to the external level, it should be noted that by objective situations we do not simply mean situations independent of the agent. We also include situations corresponding to propositions expressing the objective relation between an agent's various subjective propositions.

As we have already mentioned, objective relations between an agent's various beliefs are presented as an external aspect of the LSB3_1 logic. More precisely, we assume that the main role of formal systems is to supervise, to ensure correctness of formal reasoning. In other words, we want the conclusions drawn by the agent to stay in accordance with the objective relations between the premises and the conclusions of formal reasoning.

As to the internal level, we assume that one of the ways with which the agent shapes his beliefs about a given compound proposition is the analysis of his beliefs concerning the arguments of the main operator. Consequently:

- (Disj 1) if the agent is convinced that at least one of two propositions is true, he also has a strong belief that the disjunction of these propositions is true,
- (Disj 2) if the agent is not convinced that either disjunct is true, then he is also not convinced that the disjunction of these propositions is true.

Applying these assumptions in LSB3_1, we characterize internal logic connectives semantically by using truth tables for Kleene's strong logic[8].

8 S.C. Kleene, *On a notation for ordinal numbers*, op. cit., 153.

As to the atomic sentences, we assume that there are various ways the agent may come to accept them. For example, some atomic sentences can be accepted by the agent on the grounds of experience, some of them on the grounds of intuition, while others on the grounds of authority. It is obvious that subjects have different intuitions on this matter and quote others as authority. In LSB3_1, we do not differentiate between the different ways we come to accept atomic propositions, but we allow that there are not universally shared beliefs by all people (by all agents). We also assume the independence of certain objective propositions (or situations) from subjective propositions relating to the occurrence of such situations. The need to introduce this assumption seems to be obvious, since there are many examples of propositions accepted as true (or false) by a given person (or group of persons) when in fact the propositions are false (or true).

The above-mentioned assumptions are reflected by the fact that no formula expressing the agent's strong belief is a tautology of LSB3_1.

2. LSB3_1 SYNTAX

Let J be the language of classical propositional logic. The alphabet of language J consists of:

- variables: p, q, r, s, …,
- connectives: ~, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$,
- brackets: (,), [,].

Let J* be the language of LSB3_1. We obtain the alphabet of language J* by adding two operators: C, D to language J.

Let α be any formula (propositional statement) of language J. It follows that C(α) and D(α) are also formulae (propositional statements) of language J*.

The symbolic notation: C(α) is to be read as *The agent is convinced that* α, and D(α) is to be read as *The agent admits that* α.

The scope of the notion of LSB3_1 language formula is indicated by Definition 1. Let FOR be the conventionally defined set of language J formulae, and FOR* the set of language J* formulae. We then have:

Definition 1: FOR* is the smallest set fulfilling conditions:

(a) $FOR \subset FOR^*$

(b) if $\alpha \in FOR$, then: $C(\alpha) \in FOR^*$ and $D(\alpha) \in FOR^*$

(c) if $\alpha^*, \beta^* \in FOR^*$, then: $\sim\alpha^*$, $\alpha^* \wedge \beta^*$, $\alpha^* \vee \beta^*$, $\alpha^* \rightarrow \beta^*$, $\alpha^* \leftrightarrow \beta^*$, are also elements of set FOR*.

The formulae described in Definition 1.(b) are called **internal logic formulae**. Formulae in line with Definition 1.(a) and 1.(c) are called **external logic formulae.**

Definition 2:

The J formula given in brackets immediately after the operator is called the **scope** of a given operator for beliefs in formula α^*.

Some of the connectives given in formula: $\alpha^* \in FOR^*$ are called internal connectives; other connectives are called external connectives.

Definition 3: An **internal connective** in formula $\alpha^* \in FOR^*$ is a connective which occurs in the scope of formula α^* operator C, D.

Definition 4: An **external connective** of formula $\alpha^* \in FOR^*$ is a connective which does not occur in the scope of any formula α^* operator C, D.

According to the above-mentioned definitions, in formula (ex 1): $D(\sim p \vee \sim q) \leftrightarrow \sim C(p \wedge q)$, the scope of operator D is the expression: $\sim p \vee \sim q$, and the scope of operator C is the expression: $p \wedge q$. In formula (ex 1), both the first and second occurrences of a negation connective are internal connectives, because they occur within the operator D scope, whereas the third occurrence of a negation connective is an external connective of the formula, because it does not occur neither within the scope of operator D nor within the scope of operator C. The only occurrence of an equivalence connective is an external connective, whereas the only occurrence of a conjunction is an internal connective of the formula, because it occurs within the scope of operator C. It is worth mentioning here that in a formula without operators for beliefs, such as $(p \wedge q) \rightarrow r$, all connectives are external connectives.

3. A SEMANTICS FOR LSB3_1

Adjusting Bochvar's idea for dividing logic to the needs of our construction of LSB3_1, the matrix M* for language J* is expressed by:

$M^*= ((\{0, 1/2, 1\}, \neg, \cap, \cup, \Rightarrow, \Leftrightarrow, \neg, \cap, \cup, \Rightarrow, \Leftrightarrow, D), \{1\})$, where:

– 1 is a distinguished element of {0, ½, 1} set,

– operations: $\neg, \cap, \cup, \Rightarrow, \Leftrightarrow$, are semantic counterparts to external connectives,

– operations: $\neg, \cap, \cup, \Rightarrow, \Leftrightarrow$, are semantic counterparts to internal connectives,

– operation D is the semantic counterpart to operator D.

It is worth noting that operator C does not have a semantic equivalent. This results from the initial assumption imposed on LSB3_1 that the logical values of certain propositions to belief do not depend on the logical values of propositions referring to the agent's beliefs. Therefore, we will assign elements of set {0, ½, 1} directly to atomic formulae of the form: C(w), where w is a propositional variable.

The operation D: $\{0, 1/2, 1\} \rightarrow \{0, 1/2, 1\}$, designated by the following table is the semantic interpretation of operator D:

a	D(a)
0	0
½	1
1	1

Operations corresponding to external connectives are defined in line with Bochvar's idea.

Here is a semantic table for external negation:

a	¬ a
0	1
½	1
1	0

What follows are semantic tables for: external disjunction, external conjunction, external implication and external equivalence, respectively:

a∪b	0	½	1
0	0	0	1
½	0	0	1
1	1	1	1

a ∩ b	0	½	1
0	0	0	0
½	0	0	0
1	0	0	1

a⇒b	0	½	1
0	1	1	1
½	1	1	1
1	0	0	1

a⇔b	0	½	1
0	1	1	0
½	1	1	0
1	0	0	1

Operations which are semantic equivalents of internal connectives are characterized in line with Kleene's idea. Here is a semantic table for internal negation:

a	¬ a
0	1
½	½
1	0

What follows are semantic tables for: internal disjunction, internal conjunction, internal implication and internal equivalence, respectively:

a ∪b	0	½	1
0	0	½	1
½	½	½	1
1	1	1	1

a ∩ b	0	½	1
0	0	0	0
½	0	½	½
1	0	½	1

a⇒b	0	½	1
0	1	1	1
½	½	½	1
1	0	½	1

a⇔b	0	½	1
0	1	½	0
½	½	½	½
1	0	½	1

The function of valuation is defined as follows:

Definition 5:

Let v be a function such that: **v**: FOR* →{0, ½, 1}. It follows that **v** is a valuation if and only if the given conditions are satisfied:

(1) if w is a propositional variable, then:
(a) $\mathbf{v}(w) \in \{0, 1\}$, (b) $\mathbf{v}(C(w)) \in \{0, ½, 1\}$
(2) if $\alpha, \beta \in$ FOR, then:
(a) $\mathbf{v}(\ C(\sim\alpha)\ ) = \neg\ \mathbf{v}(\ C(\alpha)\ )$, (b) $\mathbf{v}(\ C(\alpha \wedge \beta)\ ) = \mathbf{v}(\ C(\alpha)\ ) \cap \mathbf{v}(\ C(\beta)\ )$,
(c) $\mathbf{v}(\ C(\alpha \vee \beta)\ ) = \mathbf{v}(\ C(\alpha)\ ) \cup \mathbf{v}(\ C(\beta)\ )$, (d) $\mathbf{v}(\ C(\alpha \to \beta)\ ) = \mathbf{v}(\ C(\alpha)\ ) \Rightarrow \mathbf{v}(\ C(\beta)\ )$
(e) $\mathbf{v}(\ C(\alpha \leftrightarrow \beta)\ ) = \mathbf{v}\ (\ C(\alpha)\ ) \Leftrightarrow v(\ C(\beta)\ )$
(3) if $\alpha^*, \beta^* \in$ FOR*, then:
(a) $\mathbf{v}(\sim\alpha^*\ ) = \neg\ \mathbf{v}(\ \alpha^*\ )$, (b) $\mathbf{v}(\ \alpha^* \wedge \beta^*)\ ) = \mathbf{v}(\ \alpha^*\ ) \cap \mathbf{v}(\ \beta^*\ )$,
(c) $\mathbf{v}(\alpha^* \vee \beta^*) = \mathbf{v}(\alpha^*\ ) \cup \mathbf{v}(\beta^*)$, (d) $\mathbf{v}(\alpha^* \to \beta^*) = \mathbf{v}(\alpha^*\ ) \Rightarrow \mathbf{v}(\beta^*)$,
(e) $\mathbf{v}(\alpha^* \leftrightarrow \beta^*\ ) = \mathbf{v}(\alpha^*) \Leftrightarrow \mathbf{v}(\beta^*)$
(4) if $\alpha \in$ FOR, then: $\mathbf{v}(\ D(\alpha)\ ) = \neg\ \neg\ \mathbf{v}(\ C(\alpha)\ )$

First of all, it seems that point 1 of the above definition 5 needs explaining. In this regard, earlier investigations with E. Nieznański[9] showed that propositions expressing the occurrence or absence of situations are governed by different principles than propositions expressing the agent's opinion about the occurrence and absence of objective situations. In general, the logic concerning objective situations is, in my opinion, a two-valued logic. In definition 5 this is expressed by limiting the set of values of propositional variables to the set {0,1} (point (1a) of definition 5) and characterizing external connectives according to the tables for the external part of Bochvar's logic (point (3) of definition 5). With regard to propositions expressing the agent's opinion, I allow three fundamentally different cognitive situations: 1) a strong belief that the proposition is a fact; 2) a strong belief that the proposition is not a fact; and 3) lack of a strong belief whether a proposition is a fact or not. This is precisely the reason why the set of values assigned to propositions expressing an agent's beliefs is three-valued. It is equally important to stress that I reject

9 J. Wesserling, E. Nieznański, *On the concept of truth in an intended model of the logic of beliefs*, Studia Philosophiae Christianae 49(2013)1, 135–149.

the assumption of the cognitive infallibility of an agent. In LSB3_1 logic this rejection takes the form of the independence of sentences replacing propositional variables from sentences expressing beliefs about them. This is because in structure M* the semantic equivalent of operator of belief C has not been introduced and the values of atomic formulas of C(w) form have been arbitrarily assigned. Based on point (1) of definition 5, the following cases are possible:

v(p)	v(C(p))
0	0
0	½
0	1
1	0
1	½
1	1

Next, based on point (2) of definition 5 we can state that the logical values of propositions expressing an agent's beliefs about complex sentences are dependent only on the agent's *beliefs* about simple sentences, but not on the objective sentences themselves. Thus, for instance, if $v(C(p))=1$ and $v(C(q))= ½$, then $v(C(p\wedge{\sim}q) = v(C(p)) \cap v(C({\sim}q)) = v(C(p)) \cap \neg v(C(q)) = 1 \cap (\neg ½) = 1 \cap ½ = ½$. In other words, regardless of the logical value of the proposition of form $p\wedge{\sim}q$, if an agent holds a strong belief that the former proposition is a fact and he does not hold a strong belief about the latter proposition, then he does not hold a strong belief about the logical value of the negation of the latter proposition. Hence, he does not hold a belief about the conjunction of p and ~q either.

Due to element 1 in matrix M*, the terms: *is satisfied* by a given logical valuation and *tautology*, are defined as usual:

Formula α^* (where $\alpha^* \in$ FOR*) is satisfied by **v** if and only if **v**(α^*)=1.

Formula α^* (where $\alpha^* \in$ FOR*) is a tautology of LSB3_1 if and only if α^* is satisfied under every valuation **v**.

Many of LSB3_1 tautologies are obtained on the basis of the following theorem:

Theorem 1: Let $\alpha \in$ FOR. Then, if α is a tautology of CPL, then formula α^* obtained from α by replacing propositional variables with any given formulae of the language of LSB3_1, is an LSB3_1 tautology.

In other words, all tautologies of classical propositional logic expressed in the LSB3_1 language are LSB3_1 tautologies.

Sketch of the proof:

If all propositional variables of formula α (where $\alpha \in$ FOR) are replaced by formulae of LSB3_1, then all connectives of the α formula become external connectives in the α^* formula. The property given in the theorem follows from the classical (0/1) characteristic of external connectives (according to Bochvar's three-valued logic).

The formulae obtained from the following schemata are other examples of tautologies of LSB3_1:

1. $C(\alpha) \rightarrow \sim C(\sim\alpha)$,
2. $C(\alpha \wedge \beta) \leftrightarrow (C(\alpha) \wedge C(\beta))$,
3. $C(\alpha) \vee \sim C(\alpha)$,
4. $\sim C(\alpha \wedge \sim \alpha)$,
5. $C(\alpha) \leftrightarrow \sim D(\sim\alpha)$,
6. $D(\alpha) \leftrightarrow \sim C(\sim\alpha)$,
7. $(C(\alpha \rightarrow \beta) \wedge C(\alpha)) \rightarrow C(\beta)$,
8. $D(\alpha) \vee D(\sim\alpha)$, (9) $D(\alpha \vee \sim\alpha)$, (10)$C(\alpha) \leftrightarrow C(\sim \sim\alpha)$.

It is worth noting that formulae (1) and (4) express the consistency of strong beliefs. Formula (7) is an epistemic counterpart to modus ponendo ponens. Formulae (1), (4) and (7) express the relevant assumptions concerning the so-called rational agent. Formulae (5) and (6) express the relation between strong belief and admission.

A certain set of non-tautological formulae from FOR* is designated by the following theorem.

Theorem 2:

Let $\alpha, \beta \in$ FOR, let $\& \in \{\sim, \wedge, \vee, \rightarrow, \leftrightarrow\}$. It follows that, if formula: $\alpha \& \beta$ is not a tautology of CPL, then formula: $C(\alpha) \& C(\beta)$ is not an LSB3_1 tautology.

Proof:

Let us assume that formula $\alpha \& \beta$ is not a CPL tautology. There exists, then, a classical valuation **v**: FOR $\rightarrow \{0, 1\}$ such that $v(\alpha \& \beta) = 0$. Let $w_1, w_2, \ldots, w_n$, be all propositional variables occurring in formula $\alpha\&\beta$. Let $\mathbf{v}(w_1), \mathbf{v}(w_2), \ldots, \mathbf{v}(w_n)$ be classical logical values assigned to propositional variables by logical valuation **v**. Let **v'**: FOR* $\rightarrow \{0, ½, 1\}$ be a valuation such that $\mathbf{v}'(w_1) = \mathbf{v}(w_1)$, $\mathbf{v}'(w_2) = \mathbf{v}(w_2), \ldots, \mathbf{v}'(w_n) = \mathbf{v}(w_n)$. Note that formulae $C(\alpha)$, $C(\beta)$ are LSB3_1 internal formulae. Because of this fact and the relevance of semantic tables for internal connectives in set {0,1} to semantic tables for corresponding connectives of CPL, we obtain: $\mathbf{v}'(C(\alpha)) = \mathbf{v}(\alpha)$ and $\mathbf{v}'(C(\beta)) = \mathbf{v}(\beta)$.

Therefore, due to the fact that connective & is external to formula: $C(\alpha)\& C(\beta)$, we obtain: $\mathbf{v}'(C(\alpha) \& C(\beta)) = \mathbf{v}'(C(\alpha)) \& \mathbf{v}'(C(\beta)) = \mathbf{v}(\alpha) \& \mathbf{v}(\beta) = \mathbf{v}(\alpha \& \beta) = 0$.

Thus, formula: $C(\alpha) \& C(\beta)$ is not an LSB3_1 tautology.

Obviously, this theorem does not comprise all LSB3_1 formulae that are not LSB3_1 tautologies.

For instance, LSB3_1 non-tautological formulae that are not suggested by Theorem 2, are the following:

11. $\sim C(\sim p) \rightarrow C(p)$,
12. $C(p) \vee C(\sim p)$
13. $C(p)$
14. $(C(p) \leftrightarrow C(q)) \rightarrow (C(\sim p) \leftrightarrow C(\sim q))$.

The fact that (11) and (12) are not tautologies reflects the intuition according to which the agent cannot have a strong belief either that a proposition p is true, or that a proposition denoted by the formula ~p is true. Let us note that, according to (13), none of the atomic formulae of an agent's strong belief is an LSB3_1 tautology. Formula

(14) is of significant importance. It is not a tautology due to logical valuation v(C(p)) = 0, v(C(q)) = ½, which results in the following:
v ((C(p) $\leftrightarrow$ C(q)) $\rightarrow$ (C(~p) $\leftrightarrow$ C(~q))) = (0 $\Leftrightarrow$ ½) $\Rightarrow$ (1 $\Leftrightarrow$ ½) = 1 $\Rightarrow$ 0 = 0.

However, LSB3_1 has its property stated in the following theorem:

Theorem 3:

If, for a given logical valuation **v**, the following equations hold: (1) **v**(C(α)$\leftrightarrow$C(β))=1 and (2) **v**(C(~α) $\leftrightarrow$ C(~β))=1, then **v**(C(α)) = **v**(C(β)).

Proof:

Let us assume that: (1) **v**(C(α) $\leftrightarrow$ C(β)) =1, and (2) v(C(~α) $\leftrightarrow$ C(~β)) =1. Given (1) and the table for operation $\Leftrightarrow$, the following cases are possible:

(a1) **v**(C(α)) = **v**(C(β)) = 1,

(a2) **v**(C(α)) $\neq$ 1 and **v**(C(β)) $\neq$ 1

In case (a1), we obtain: v(C(α)) = v(C(β)).

In case (a2), there are two possibilities:

(a2.1) v(C(α)) = 0. Then, v(C(~α)) =1.With respect to (2), we have: v(C(~β)) =1. Therefore, v(C(β)) = v(C(~(~ β)) = 0, which consequently gives us the following: v(C(α)) = v(C(β)).

(a2.2) v(C(α)) = ½. If v(C(β)) = 0, then v(C(~α)) = ½ and v(C(~β)) =1. Therefore, v(C(~α) $\leftrightarrow$ C(~β)) = ½$\Leftrightarrow$1 = 0, which contradicts assumption (2). That being so, if v(C(α)) = ½, then it is also the case that v(C(β)) = ½. Hence, v(C(α)) = v(C(β)).

There are no other cases to be considered, for every case v(C(α))= v(C(β)). The proof has therefore been completed.

Let us conclude this section by highlighting certain limitations of LSB3_1 with regard to other logics of belief. One of the limitations concerns the kind of beliefs the agent's subjective proposition refers to. More precisely, in LSB3_1 we deal with subjective propositions referring only to situations external to the agent. We do not, therefore, take into consideration introspection and self-awareness of the agent. Generally, it is assumed that the agent is convinced that α if the agent is convinced that he is convinced that α. In LSB3_1, this condition is strengthened to equivalence. In other words, if we admitted the self-awareness of a rational agent in LSB3_1, every formula of the form $C(C(\alpha)) \leftrightarrow C(\alpha)$ would be a tautology. Using "Occam's Razor", and through definition 1, we limit the set FOR*. The second limitation will be introduced in section 4. Because we want to apply Kleene's strong logic language (without any belief operators) to J* in order to compare the two logics, only formulae concerning an agent's subjective propositions will be connected by external connectives. Therefore, the set FOR* will not include, e.g., expressions of the following forms:

15. $\alpha \vee {\sim}\alpha$,
16. $C(\alpha) \rightarrow \alpha$,
17. $(\alpha \wedge \beta) \leftrightarrow (D(\alpha) \wedge D(\beta))$, where $\alpha, \beta \in$ FOR.

Recalling our initial assumption of the independency of an agent's subjective propositions from objective ones, to which they refer, it seems that the above limitation will not weaken our considerations.

4. THE AXIOMATIZATION OF LSB3_1

In light of the remarks in the final part of section 3, we limit the set FOR* of LSB3_1 according to the following definition:

Definition 6: FOR* is the smallest set fulfilling conditions: (a) if $\alpha \in$ FOR , then: $C(\alpha) \in$ FOR* and $D(\alpha) \in$ FOR* , and (b) if $\alpha^*, \beta^* \in$ FOR* , then: ${\sim}\alpha^*$, $\alpha^* \wedge \beta^*$, $\alpha^* \vee \beta^*$, $\alpha^* \rightarrow \beta^*$. $\alpha^* \leftrightarrow \beta^*$ are also elements of set FOR*.

Urquhart (2001) formulates the axiomatic approach to Kleene's strong three-valued logic (pp. 256–260). This logic comprises: 1) two definitional equations; 2) thirteen axioms; and 3) two primitive rules of inference. Using some external connectives (external conjunction and external equivalence}, whose equivalents are absent in the

Urquhart's paper, it is possible to characterize Kleene's logic in the LSB3_1 language J* by using formulae with the following schemata:

meta-definitions for two internal connectives:

(D1) $C(\alpha \rightarrow \beta) = C(\sim\alpha \vee \beta)$,

(D2) $C(\alpha \leftrightarrow \beta) = C[(\alpha \wedge \beta) \vee (\sim\alpha \wedge \sim\beta)]$

(I) schemata of axioms:

(AK1) $C(\alpha) \leftrightarrow C(\sim\sim\alpha)$,(AK3)

(AK2) $[C(\alpha) \wedge C(\sim\alpha)] \rightarrow C(\beta)$,(AK4)

(AK3) $C(\alpha \wedge \beta) \leftrightarrow (C(\alpha) \wedge C(\beta))$, (AK5)

(AK4) $C(\alpha) \rightarrow C(\alpha \vee \beta)$, (AK6)

(AK5) $C(\beta) \rightarrow C(\alpha \vee \beta)$, (AK7)

(AK6) $C(\sim\alpha) \rightarrow C(\sim(\alpha \wedge \beta))$, (AK8)

(AK7) $C(\sim\beta) \rightarrow C(\sim(\alpha \wedge \beta))$, (AK9)

(AK8) $(C(\sim\alpha) \wedge C(\sim\beta)) \rightarrow C(\sim(\alpha \vee \beta))$, (AK10)

(AK9) $C(\sim(\alpha \wedge \beta)) \rightarrow C(\sim\alpha \vee \sim\beta)$, (AK11)

(AK10) $C(\sim(\alpha \vee \beta)) \rightarrow C(\sim\alpha \wedge \sim\beta)$, (AK12)

The only primitive rules of inference in the [8] version are: the Cut Rule (R_C) and the Dilemma Rule (R_D). It seems that the meta-rules can be characterized as follows:

Let $\Gamma \subset FOR$, $\Delta \subset FOR$ and $C(\Gamma) = \{C(\gamma): \gamma \in \Gamma\}$, $C(\Delta) = \{C(\delta): \delta \in \Gamma\}$. Then:

(R_C) if $C(\Gamma), C(\alpha) \vdash C(\beta)$ and $C(\Delta) \vdash C(\alpha)$, then $C(\Gamma), C(\Delta) \vdash C(\beta)$

(R_D) if $C(\Gamma), C(\alpha) \vdash C(\varepsilon)$ and $C(\Gamma), C(\beta) \vdash C(\varepsilon)$, then $C(\Gamma), C(\alpha \vee \beta) \vdash C(\varepsilon)$

This axiomatic account is too weak for LSB3_1. It results from: 1) the lack of equivalents of the external disjunction connective and the external negation connective in Kleene's higher axiomatized logic; and 2) a different (weaker) understanding of the internal disjunction connective equivalent with respect to the one employed in LSB3_1 semantics.

We shall now construct the calculus for LSB3_1 by giving axiom schemata and one primitive rule of inference. The axioms are divided into two groups: **(A)** axioms for external connectives, and **(B)** axioms for internal connectives.

Let, $\alpha, \beta \in FOR$, $\alpha^*, \beta^*, \gamma^* \in \underline{FOR}^*$.

(A) $\alpha^* \to (\beta^* \to \alpha^*)$

$[\alpha^* \to (\beta^* \to \gamma^*)] \to [(\alpha^* \to \beta^*) \to (\alpha^* \to \gamma^*)]$

$(\alpha^* \to \beta^*) \to (\sim\beta^* \to \sim\alpha^*)$

$\sim(\sim\alpha^*) \to \alpha^*$

$\alpha^* \to \sim(\sim\alpha^*)$

$(\alpha^* \wedge \beta^*) \to \alpha^*$

$(\alpha^* \wedge \beta^*) \to \beta^*$

$(\alpha^* \to \beta^*) \to \{(\alpha^* \to \gamma^*) \to [\alpha^* \to (\beta^* \wedge \gamma^*)]\}$

$\alpha^* \to (\alpha^* \vee \beta^*)$

$\beta^* \to (\alpha^* \vee \beta^*)$

$(\alpha^* \to \gamma^*) \to \{(\beta^* \to \gamma^*) \to [(\alpha^* \vee \beta^*) \to \gamma^*]\}$

$(\alpha^* \leftrightarrow \beta^*) \to (\alpha^* \to \beta^*)$

$(\alpha^* \leftrightarrow \beta^*) \to (\beta^* \to \alpha^*)$

$(\alpha^* \to \beta^*) \to [(\beta^* \to \alpha^*) \to (\alpha^* \leftrightarrow \beta^*)]$

(B) I introduce here three meta-definitions and six axiom schemata:

(MDef 1) $C(\alpha \to \beta) = C(\sim\alpha \vee \beta)$,

(MDef 2) $C(\alpha \leftrightarrow \beta) = C[(\alpha \wedge \beta) \vee (\sim\alpha \wedge \sim\beta)]$

(MDef 3): $D(\alpha) = \sim C(\sim\alpha)$

(Ax 1) $C(\alpha) \leftrightarrow C(\sim\sim\alpha)$,

(Ax 2) $C(\alpha \wedge \beta) \leftrightarrow (C(\alpha) \wedge C(\beta))$,

(Ax 3) $C(\alpha \vee \beta) \leftrightarrow (C(\alpha) \vee C(\beta))$,

(Ax 4) $C(\sim(\alpha \wedge \beta)) \leftrightarrow C(\sim\alpha \vee \sim\beta)$,

(Ax 5) $C(\sim(\alpha \vee \beta)) \leftrightarrow C(\sim\alpha \wedge \sim\beta)$

(Ax 6) $\sim C(\alpha \wedge \sim\alpha)$

The only primitive rule of inference in LSB3_1 is the modus ponens rule (MP-rule):

$$\alpha^*, \alpha^* \to \beta^* \vdash \beta^*.$$

According to the axiom schemata in group **(A)** and the MP-rule, LSB3_1 is the extension of classical propositional calculus in the sense that all classical propositional calculus theses expressed in LSB3_1 are LSB3_1 theses and they are all valid. Among others, the following inference rules are valid in LSB3_1:

OA1 : $\sim \alpha^* \vee \beta^*,\ \alpha^* \vdash \beta^*$
OA2: $\alpha^* \vee \sim \beta^*, \beta^* \vdash \alpha^*$
DA1 : $\alpha^* \vdash \alpha^* \vee \beta^*$
DA2: $\beta^* \vdash \alpha^* \vee \beta^*$
OK1 : $\alpha^* \wedge \beta^* \vdash \alpha^*$
OK2: $\alpha^* \wedge \beta^* \vdash \beta^*$
DK : $\alpha^*, \beta^* \vdash \alpha^* \wedge \beta^*$
OR1 : $\alpha^* \leftrightarrow \beta^* \vdash \alpha^* \rightarrow \beta^*$
OR2: $\alpha^* \leftrightarrow \beta^* \vdash \beta^* \rightarrow \alpha^*$
DR: $\alpha^* \rightarrow \beta^*, \beta^* \rightarrow \alpha^* \vdash \alpha^* \leftrightarrow \beta^*$

The following theorem holds in LSB3_1:

Natural deduction theorem:

Let $\Gamma^* \subset$ FOR * and $\alpha^*, \beta^* \in$ FOR *. Then: 1) $\Gamma^* \cup \{\alpha^*\} \vdash \beta^*$ if and only if $\Gamma^* \vdash \alpha^* \rightarrow \beta^*$, 2) $\Gamma^* \vdash \alpha^*$ if and only if there exists β^* such that $\Gamma^* \cup \{\sim\alpha^*\} \vdash \{\beta^*, \sim\beta^*\}$

With respect to the natural deduction theorem, we can introduce rules for creating direct conditional proofs and indirect conditional proofs.

I shall conclude this section by suggesting that the definitional equations (MDef 1) and (MDef 2) can be replaced by axioms of schemata: (Ax 7) – (Ax 11), where:

(Ax 7) $C(\alpha \rightarrow \beta) \leftrightarrow C(\sim\alpha \vee \beta)$
(Ax 8) $C(\sim(\alpha \rightarrow \beta)) \leftrightarrow C(\alpha \wedge \sim\beta)$
(Ax 9) $C(\alpha \leftrightarrow \beta) \leftrightarrow C[(\alpha \wedge \beta) \vee (\sim\alpha \wedge \sim\beta)]$
(Ax 10) $C(\sim(\alpha \leftrightarrow \beta)) \leftrightarrow C[(\alpha \vee \beta) \wedge (\sim\alpha \vee \sim\beta)]$
(Ax 11) $[(C(\alpha) \leftrightarrow C(\beta)) \wedge (C(\sim\alpha) \leftrightarrow C(\sim\beta))] \wedge [(C(\gamma) \leftrightarrow C(\delta)) \wedge (C(\sim\gamma) \leftrightarrow C(\sim\delta))] \rightarrow (C(\alpha \& \gamma) \leftrightarrow C(\beta \& \delta)$, where $\& \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

5. THE CONJUNCTIVE NORMAL FORM AND THE COMPLETENESS THEOREM

Before moving on to the Conjunctive Normal Form Theorem, I shall clarify some lemmas and auxiliary theorems.

Lemma 1: Let $\phi(\alpha)$ be any formula of classical propositional logic, and α a particular formula of classical propositional logic. Let $\phi(\alpha //{\sim}{\sim}\alpha)$ be the formula of classical propositional calculus obtained by replacing $\alpha //{\sim}{\sim}\alpha$ in ϕ. We then have that formula $C(\phi(\alpha)) \leftrightarrow C(\phi(\alpha //{\sim}{\sim}\alpha))$ is the LSB3_1 thesis.

Definition 7:

(I) Formula α is the elementary disjunction of language J if and only if α takes the form of $\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n$, in which, for every $i \in \{1, 2, 3, \ldots, n\}$, α_i has one the form of either (1) w or (2) ~w, where w is a propositional variable.

(II) Formula α^* is the elementary disjunction of language J* if and only if α^* takes the form of: $\alpha_1^* \vee \alpha_2^* \vee \ldots \vee \alpha_n^*$, in which, for every $i \in \{1, 2, 3, \ldots, n\}$, α_i^* has one of the following forms:

(1) C(w), (2) ~C(w), (3) C(~w), or (4) ~C(~w), where w is a propositional variable

Definition 8:

Let $\alpha^* \in \underline{FOR}^*$. Formula α^* is a formula of the conjunctive normal form if and only if it consists of finite conjunctions whose arguments are the only elementary disjunctions of language J*.

Theorem 4:

For every formula $\alpha^* \in \underline{FOR}^*$, there exists a formula $\beta_1^* \wedge \beta_2^* \wedge \ldots \wedge \beta_n^*$ such that, for every $i \in (1, 2, \ldots, n\}$, β_i^* is the disjunction of the formulae whose main operator is either C or the negation of the formula whose main operator is C.

The proof is identical to the proof in CPL because all theses of classical propositional calculus are LSB3_1 theses.

Notation: For n, m – natural numbers:

Let $\bigcap_{i=1}^{n} \alpha i$ be the abbreviation for: $\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n$

Let $\bigcup_{j=1}^{m} \alpha j$ be the abbreviation for: $\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_m$

Lemma 2: Let ω_{ji} be a propositional variable or the negation of propositional variables, for every j_i such that $j_i \in \{1, 2, \ldots, m_i\}$ and $i \in \{1, 2, \ldots, n\}$. It follows that an LSB3_1 thesis is every formula of the following form: $C\,(\bigcap_{i=1}^{n}(\bigcup_{j=1}^{mi} \omega ji) \leftrightarrow \bigcap_{i=1}^{n}(\bigcup_{j=1}^{mi} C(\omega ji))$.

This lemma results from axioms (Ax1), (Ax 2) and (Ax 3). The proof here is inductive – firstly with respect to the "length" of the elementary disjunction, and secondly with respect to the number of elementary disjunctions.

Corollary 1: If α is a formula of the conjunctive normal form in CPL, then formula C(α) is equivalent, on the basis of LSB3_1, to a given formula α^* of the conjunctive normal form in LSB3_1.

Formulae of the C(ω) form, in which ω is a propositional variable or the negation of a propositional variable, are the only arguments for elementary disjunctions in formula α^*.

Lemma 3: If α and β are formulae of the conjunctive normal form in classical propositional logic, then, on the basis of LSB3_1, the following are equivalentially reducible to the formulae of the normal form:

(1) $C(\alpha \wedge \beta)$ (2) $C(\alpha \vee \beta)$ (3) $C(\sim\alpha)$

Proof:

Let us assume that $\alpha := \bigcap_{i=1}^{n}(\bigcup_{j=1}^{mi} \omega ji)$ and $\beta := \bigcap_{k=1}^{s}(\bigcup_{l=1}^{tk} \upsilon lk)$, in which ω_{ji}, υ_{lk} are propositional variables or the negation of propositional variables (for every natural numbers $j_i \in \{1, 2, \ldots, m_i\}$, $i \in \{1, 2, \ldots, n\}$, $l_k \in \{1, 2, \ldots, t_k\}$, $k \in \{1, 2, \ldots, s\}$).

We then obtain:

(1) With respect to (Ax 2):

$$C((\bigcap_{i=1}^{n}(\bigcup_{j=1}^{mi} \omega ji)) \wedge (\bigcap_{k=1}^{s}(\bigcup_{l=1}^{tk} \upsilon lk))) \leftrightarrow [C(\bigcap_{i=1}^{n}(\bigcup_{j=1}^{mi} \omega ji)) \wedge$$
$$\wedge C(\bigcap_{k=1}^{s}(\bigcup_{l=1}^{tk} \upsilon lk))]$$

is an LSB3_1 thesis.

With respect to lemma 2:

$$[C(\bigcap_{i=1}^{n}(\bigcup_{j=1}^{mi} \omega ji)) \wedge C(\bigcap_{k=1}^{s}(\bigcup_{l=1}^{tk} \upsilon lk))] \leftrightarrow [(\bigcap_{i=1}^{n}(\bigcup_{j=1}^{mi} C(\omega ji)) \wedge$$
$$\wedge (\bigcap_{k=1}^{s}(\bigcup_{l=1}^{tk} C(\upsilon lk))]$$ is also an LSB3_1 thesis.

The right-hand side of the equivalence above is a formula of the conjunctive normal form in LSB3_1. Therefore, because the external equivalence connective is transitive,

$C(\alpha \wedge \beta)$ is also equivalentially reducible to the formulae of the normal form.

(2) with respect to (Ax3):

$$C((\bigcap_{i=1}^{n}(\bigcup_{j=1}^{mi} \omega ji)) \vee (\bigcap_{k=1}^{s}(\bigcup_{l=1}^{tk} \upsilon lk))) \leftrightarrow [C(\bigcap_{i=1}^{n}(\bigcup_{j=1}^{mi} \omega ji)) \vee$$
$$\vee C(\bigcap_{k=1}^{s}(\bigcup_{l=1}^{tk} \upsilon lk))]$$

is an LSB3_1 thesis.

With respect to lemma 2:

$$[C(\bigcap_{i=1}^{n}(\bigcup_{j=1}^{mi} \omega ji)) \vee C(\bigcap_{k=1}^{s}(\bigcup_{l=1}^{tk} \upsilon lk))] \leftrightarrow [(\bigcap_{i=1}^{n}(\bigcup_{j=1}^{mi} C(\omega ji)) \vee$$
$$\vee (\bigcap_{k=1}^{s}(\bigcup_{l=1}^{tk} C(\upsilon lk))]$$ is also a thesis.

Applying the associativity of external conjunction, the distributivity of external disjunction over external conjunction, and the transitivity of the external equivalence connective, $C(\alpha \vee \beta)$ becomes equivalentially reducible to the conjunctive normal form in the LSB3_1 language.

By induction, this result is generalized into the disjunction of the n-formulae: $\alpha 1, \alpha 2, \ldots, \alpha n \in$ FOR (where n is a natural number) of the conjunctive normal forms in CPL.

(2) With respect to (Ax 4) i (Ax 5), we obtain a formula which constitutes the LSB3_1 thesis:

$C(\sim(\bigcap_{i=1}^{n}(\bigcup_{j=1}^{mi}\omega ji)) \leftrightarrow C((\bigcup_{i=1}^{n}(\bigcap_{j=1}^{mi}\sim\omega ji))$.

With respect to lemma 2, $C(\bigcup_{i=1}^{n}(\bigcap_{j=1}^{mi}\sim\omega ji)) \leftrightarrow ((\bigcup_{i=1}^{n}(\bigcap_{j=1}^{mi}C(\sim\omega ji)))$ is also a thesis.

On the basis of LSB3_1, the generalization in point (2) of this proof and axiom (Ax1), the right-hand side of the proof is equivalent to the formula of the conjunctive normal form in which propositional variables or the negations of propositional variable are arguments of elementary disjunctions.

Lemma 4: For all formulae α of classical propositional calculus which have no connectives other than: the negation connective, the conjunction connective and the disjunction connective, there exists a formula β* of the conjunctive normal form in language J* such that formula: C(α) ↔ β* is a thesis of LSB3_1.
Inductive proof relative to the number of connectives in formula α.

Corollary 2: Every formula of the C(α) form is equivalentially reducible to the conjunctive normal form.

Theorem 6 (about the reduction to the conjunctive normal form in LSB3_1)**:**

Let J* be the language of LSB3_1 and α* ∈ FOR*. Formula α* can then be equivalentially reducible to the conjunctive normal form, in which only formulae of the following forms:

1) C(w),

2) C(~w),

3) ~C(w), and

4) ~C(~w), where w is a propositional variable, can be the arguments of elementary disjunctions.

Proof:

With respect to Theorem 4, Corollary 2 and the extentionality of external connectives, it is sufficient to point out that every formula of the ~ C(α) form is equivalentially reducible to the conjunctive normal form. Also, from Corollary 2 follow two further facts: (1) there exists a formula of the following form: $\bigcap_{i=1}^{n}(\bigcup_{j=1}^{mi} C(\omega ji))$, where ω_{ji} is a propositional variable or its negation, and (2) formula:

$C(\alpha) \leftrightarrow (\bigcap_{i=1}^{n}(\bigcup_{j=1}^{mi} C(\omega ji)))$ is an LSB3_1 thesis.

We then obtain that: $\sim C(\alpha) \leftrightarrow \sim (\bigcap_{i=1}^{n}(\bigcup_{j=1}^{mi} C(\omega ji)))$ is also a thesis.

The right-hand side of this equivalence is reducible to the conjunctive normal form on the basis of: (1) De Morgan's Laws of external conjunction and external disjunction, (2) the association of external conjunction and the association of external disjunction, and (3) the distribution of external conjunction over external disjunction.

Thereofore, $\sim C(\alpha)$ is equivalentially reducible to the conjunctive normal form in LSB3_1.

The theorem mentioned above will be used as proof of the Completeness Theorem.

Weak Completeness Theorem:

Let J* be the language of LSB3_1. Let $\alpha^* \in$ <u>FOR</u>*. It follows that α^* is the LSB3_1 thesis if and only if α^* is the tautology of LSB3_1.

$\rightarrow$) Definitions (MDef 1), (MDef 2) and (MDef D) are tautologies because, on the basis of internal connectives tables, for each a, b $\in$ {0, ½, 1} the following equations hold:

1. $(a \Rightarrow b) = [(\neg a) \cup b]$,
2. $(a \Leftrightarrow b) = [(a \cap b) \cup ((\neg a) \cap (\neg b))]$
3. $D(a) = \neg(\neg a)$

Thus, the tautological character of axioms (Ax1) $C(\alpha) \leftrightarrow C(\sim\sim\alpha)$, (Ax4) $C(\sim(\alpha \wedge \beta)) \leftrightarrow C(\sim\alpha \vee \sim\beta)$, (Ax5) $C(\sim(\alpha \vee \beta)) \leftrightarrow C(\sim\alpha \wedge \sim\beta)$ follows from equations:

4. $a = \neg(\neg a)$
5. $\neg \min\{a, b\} = \max\{\neg a, \neg b\}$, where min of two values is equal to less of the two values (in the arithmetical sense) and max of two values is equal to bigger of the two values (in the arithmetical sense), and
6. $\neg \max\{a, b\} = \min\{\neg a, \neg b\}$

The tautological character of (Ax 2) $C(\alpha \wedge \beta) \leftrightarrow (C(\alpha) \wedge C(\beta))$ follows from the following facts:

7. min {a, b} ≠ 1 if and only if: a ≠ 1 and b ≠ 1. However, if min {a, b} ≠ 1, then $a \cap b$ =0 and ($a \cap b$ = 0 or $a \cap b$ =½). But (0 ⇔ 0) =1 and (½ ⇔ 0)=1=, thus [($a \cap b$) ⇔ ($a \cap b$)]=1 . If min {a, b} = 1, then a = b = 1 and ($a \cap b$) ($a \cap b$) = 1, thus [($a \cap b$) ⇔ ($a \cap b$)] = (1 ⇔ 1) =1.
8. It can be proven analogically that it is always the case that: [($a \cup b$) ⇔ ($a \cup b$)] = 1. That is, (Ax 3) (Ax 3) $C(\alpha \vee \beta) \leftrightarrow (C(\alpha) \vee C(\beta))$ is an LSB3_1 tautology.

Lastly, the tautological character of (Ax 6) $\sim C(\alpha \wedge \sim \alpha)$ follows from the fact that $a \cap \neg a \neq 1$. Thus, $\neg(a \cap \neg a)$ =1.

The validity of the MP-rule: $\alpha^*, \alpha^* \rightarrow \beta^* \vdash \beta^*$, which is the only primitive role of inference in LSB3_1, follows from the fact that the rule applies only to external connectives.

←) On account of the equivalential reducibility of the formulae of language J* of LSB3_1 to the conjunctive normal form, it is sufficient to prove that: if α^* is the formula of the conjunctive normal form and the tautology of LSB3_1, then α^* is the LSB3_1 thesis in the aforementioned axiomatic perspective.

Thus, let us assume that α^* is the LSB3_1 tautology of the conjunctive normal form in LSB3_1.

It then follows that all elementary disjunctions which are arguments for (external) conjunction connectives must also be tautologies.

However, the elementary disjunction above is the LSB3_1 tautology if and only if it has:

1. C(w) and ~C(w), or
2. ~C(w) and ~C(~w), or
3. C(~w) and ~C(~w),

where w is a propositional variable.

Thus, every formula of the following forms:

1'. $C(w) \vee \sim C(w)$,

2'. $\sim C(w) \vee \sim C(\sim w)$, and

3'. $C(\sim w) \vee \sim C(\sim w)$

is an LSB3_1 thesis.

Therefore, with respect to rules (DA) and (DK), we conclude that α^* can be given a proof in LSB3_1.

6. PROSPECTS FOR FURTHER RESEARCH

The LSB3_1 system described here relies on the intuition that an agent shapes his strong belief on a given compound proposition on the basis of his beliefs concerning its constituents. We did not take into account the fact that sometimes an agent has a strong belief that a given compound proposition is true (or that it is false), even if he does not have a strong belief on the logical value of any of its propositional constituents. I intend to explore these possibilities in further developments of a logic of strong belief based on Kleene's and Bochvar's logics.

Another direction for future research concerns the number of operators of subjective attitudes. LSB3_1 deals only with two doxastic operators: 1) strong belief (C), and 2) admission (D).

The set of belief operators can be extended by adding, e.g., the operator of weak belief. This would result in an increased number of logical values in a certain logic of beliefs. It is also worth considering that there are propositions which are not taken under consideration by the agent, because they do not exist in the agent's consciousness. Logics that take this possibility into account will probably impose some limitations on the rule of replacement.

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